INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng 2006; **65**:1221–1242
Published online 19 September 2005 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/nme.1485

Stochastic analysis of highly non-linear structures

Willem Roux[1,*,†], Nielen Stander[1], Frank Günther[2] and Heiner Müllerschön[3]

[1]*Livermore Software Technology Corporation, 7374 Las Positas Road, Livermore, CA 94550, U.S.A.*
[2]*DaimlerChrysler AG, Germany*
[3]*DYNAmore GmbH, Germany*

SUMMARY

Instabilities can introduce highly non-linear effects into structural problems. The instabilities, not clearly associated with a change in a parameter, result in a stochastic variation of the responses. This process variation can be distinguished from the effects of the parameter variation by mapping the response variation onto a predictable space and a residual space, where the predictable space contains the possible effects of the parameter variation, and the residual space contains the process effects. This study discusses the sources (mechanics) of the response variation in this class of problems, the use of response surfaces to distinguish between effects driven by design variable changes and bifurcations, and the visualization of unstable zones in the structure. Analytical problems, a headform impact problem, and an occupant safety study clarify the use of the proposed methodology.

KEY WORDS: stochastic analysis; response surfaces; bifurcation analysis; stability visualization

1. INTRODUCTION

The computational results of highly non-linear structural problems—for example: vehicle crash—has a variation in the response values not associated with changes in the design variables. Due to the nature of the structural problem, bifurcations occur, resulting in multiple solution branches. This study shows how this behaviour can be considered in a non-deterministic evaluation using response surfaces.

Designing for uncertainty requires that the sources of the uncertainty be well understood. A number of sources of uncertainty apply to reliability analysis [1, 2]. The uncertainty in a problem is usually classified into reducible (epistemic) and irreducible (aleatoric) uncertainty [2]. The reducible uncertainties are a lack of knowledge or information, while irreducible

*Correspondence to: Willem Roux, Livermore Software Technology Corporation, 7374 Las Positas Road, Livermore, CA 94550, U.S.A.
†E-mail: willem@lstc.com

Contract/grant sponsor: DaimlerChrysler Commercial Vehicles

Received 3 December 2004
Revised 30 March 2005
Accepted 20 July 2005

uncertainties are known fluctuations of the model parameters. Only irreducible uncertainties are considered in this study; the uncertainty associated with the bifurcations is considered irreducible once quantified. In general, the contribution of the reducible uncertainties must be considered as well: the consideration of certification tests and conservative material properties may be crucial for the association of the theoretical safety with the actual safety [3].

For these highly non-linear structural problems, several physical solutions are considered plausible for a design [4], and it has been shown how small changes in the model result in substantial changes in the response due to the bifurcations [5]. Computationally, a high degree of precision in the results may be unobtainable, even if the program meets certain standards, due to smoothness and stability issues [6]. Of particular interest for this study is the loss of stability (some smoothness problems can be overcome using more powerful computers), because stability issues such as imperfection sensitivity and multiple equilibrium branches can dramatically amplify the variation of the parameters. This paper shows that a visual indicator of stability and smoothness problems can be obtained by plotting the residuals of the response surface fits to the nodal displacements on the computational model.

The structural response variation can accordingly be attributed to deterministic variation as well as process variation. The deterministic effects are associated with changes in the parameter values, while the process variation cannot be associated with changes in parameter values, though it can be triggered by a perturbation in a parameter value. The process variation can be due to a physical occurrence and must therefore be considered in a probabilistic evaluation. Distinguishing between the deterministic and process effects is however challenging; this paper addresses some of the aspects.

Larger assemblies of components seem to be more prone to bifurcation problems due to the problem size and a richer set of interacting eigenvalues. Simplifying a model may remove unexpected, though safety-critical, structural behaviour. The bifurcation behaviour can however be observed in simpler models; for example, the tube crush design study of Missoum *et al.* [7] where the tube was long enough to undergo crushing and occasional global buckling (a shorter tube would probably not have exhibited the occasional global buckling). Increasing the mesh density (smoothness) on the other hand, will reduce the process variation; a sufficiently fine mesh can be crucial as shown by Crisfield and Peng [8], who demonstrated that too coarse a mesh might introduce a range of artificial limit points in the non-linear analysis of a cantilevered cylinder. Similar problems occur in CFD simulations, where grid refinement can be crucial and outliers can be associated with significantly different physical behaviour [9].

The benchmark method for investigating the variability of these problems is Monte Carlo simulation. This method usually has a prohibitively high cost due to the large number of expensive simulations required. Metamodels—approximations to the structural behaviour, built using FEA evaluations of a selected set of designs—are therefore used to reduce cost. This study shows that the process variation (noise) must be incorporated (not filtered out) if the metamodel results are to be comparable to the Monte Carlo results. The current practice using metamodels is to smooth out and discard the noise in the solution [10]. This practice dates back to the formative research on using metamodels in structural design [11], in which the residuals were only examined to establish the goodness of fit of the metamodel. The motivation for smoothing out the noise is that the random error exists in physical experiments, but not in numerical experiments [10].

But well-constructed numerical experimentation should reproduce the process variation (noise) if it is a physical property of the system. The physical origin for the process variation in the example problems given is taken to be structural impact. The impact induces a broadband spectrum of initial conditions (white noise), which, in turn, leads to different bifurcation paths. To model the physical causation, random fields describing the geometric imperfections as used by Schenk and Schuëller [12] can be also used. They showed reasonably good correlation between the scatter introduced by the random imperfection fields and the scatter of experimental results for the limit load of thin-walled, cylindrical shells under axial compression.

In the following sections, the paper discusses the origins of the response variation, a method of estimating the process variation, visualization of the process variation (stability problems), and demonstrates the phenomena and methodology using example problems.

2. PROBABILISTIC BACKGROUND

2.1. Response variation

The association between the variation of the responses and the variation of the structural parameters may not always be clear, especially if a bifurcation occurred during the analysis. A change in results may be due to a change of a parameter value or a bifurcation; this section classifies the response variation in this context.

The decomposition of variation of the response is shown in Figure 1. Of particular interest is the buckling of the column shown on the right-hand side of the figure: the column can (i) buckle left at the top, (ii) buckle right at the top and hit a wall, or (iii) continue buckling locally at the bottom; all giving different results for the top displacement and energy absorption. All three cases are physically allowable, and the process variation is therefore a physical attribute of the system. The on–off impact condition shown next to the column should be seen in the context of a vehicle crash where many on–off conditions are possible, particularly in the engine compartment—multiple bifurcations in a single simulation are therefore possible.

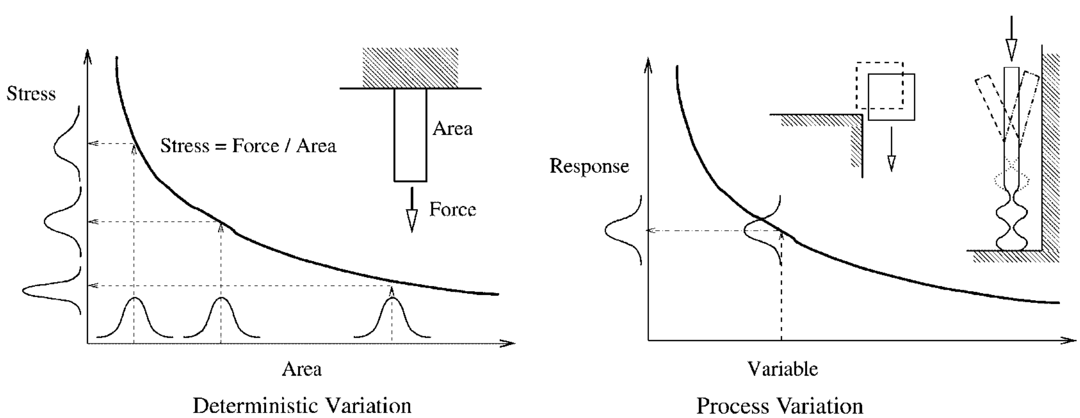

Figure 1. Decomposition of the response variation. The deterministic variation of the response is associated with a variation in the parameter value, while the process variation, caused by bifurcations and on–off impact, can occur without a change in the structural parameter value.

The response variation can therefore be decomposed as

- *Deterministic variation*: An expected, predictable, and repeatable variation in a response associated with a variation in a structural parameter. It can occur as a result of controllable or uncontrollable changes in the design. This distinction is made using the following definitions:
  - *Design variables*: These are the variation of structural parameters under the control of the designer; for example, a gauge thickness.
  - *Noise variables*: These are variations of the structure or environment not under the control of the engineer during the operation or manufacturing of the structure, but which can be controlled during analysis; for example, variation in a load.
- *Process variation*: Variation that cannot be associated with a change of a structural parameter.
  - *Physical*:
    - *Instability:* Slightly different initial conditions, especially when driven by a rich spectrum of eigenvalues, can lead to noticeable differences in responses. These different solution paths are caused by instability, and are represented in the solution mostly as bifurcations. The solution paths can be sensitive to initial values; for example, buckling initiation.
    - *On–off impact*: Slight changes in the model may cause different components to come into contact, not to come in contact, or change the order of impact.
  - *Non-physical*:
    - *Discretization*: The spatial and temporal discretizations must be fine enough that transitions occur over several increments in space and time [6, 8]. Amongst others, the following factors may cause changes in results: discretization density (mesh size), automated mesh (re)generation, and the resolution and filtering of data gathered.
    - *Numeric settings*: Iterative solvers and contact algorithm options.
    - *Different computers*: A small change in machine precision and compiler differences can lead to a large change of the response values if it triggers a (different) bifurcation.

  The physical and non-physical variations are not necessarily disjoint, and can interact; for example, a mesh change acting as a seed for a buckling mode. More sources of variability exist; see Belytschko and Mish [6] for an in-depth discussion on the various physical phenomena and related computational barriers.

The total response variation, assuming the deterministic variation and process variation are independent, can be computed as

$$\sigma^2_{\text{total}} = \sigma^2_{\text{deterministic}} + \sigma^2_{\text{process}}$$

For linear structural behaviour this typically reduces to $\sigma^2_{\text{total}} = \sigma^2_{\text{deterministic}}$, while for highly non-linear problems with fixed values of the parameters, it reduces to $\sigma^2_{\text{total}} = \sigma^2_{\text{process}}$. The replacement of homogeneous material properties and perfect geometries with stochastic fields

[12] is presumably the best method of simulating the process variation. A stochastic field is a spatial perturbation of the idealized perfect model generated using random numbers to conform to a given correlation function. Generating a number of stochastic fields allows different equilibrium branches of a structure to be explored with a Monte Carlo evaluation, with all the equilibrium branches found considered plausible for the design.

2.2. *Probabilistic evaluations*

Probabilistic methods should ideally have the following key properties for highly non-linear structural problems:

- Quantify the variation in the responses.
- Establish the link between cause and effect.
- Distinguish deterministic effects from random occurrences.

What constitutes engineering accuracy, especially at the low probabilities, is an open question. A definition such as six-sigma (a requirement for the mean and the constraint on a response to be separated by six standard deviations) may be the best way of specifying the engineering requirement; a precise numerical value may not be meaningful. The accuracy of the probabilities should however be such that different designs can be compared.

In this study, Monte Carlo simulation both with and without a metamodel is used. Much more accurate and sophisticated methods of computing reliability are available [1, 13, 14], but are outside the scope of this study. The contribution of the process variation should however be independent from the reliability method used.

Monte Carlo simulation is very expensive for the computation of small probabilities. Probabilistic computations are therefore usually combined with metamodels for which millions of function evaluations are feasible. The error of the fit of the metamodel, amongst others, contributes to the total error of computing reliability [15]. If only the mean value and the standard deviation of the response are desired, then the Monte Carlo method may be more attractive. The Monte Carlo method is actually the best method in cases where the process variation is the primary source of variation, because the use of metamodels will only contribute additional complexity and potential errors if the deterministic variation is unimportant.

3. RESPONSE SURFACE METHODOLOGY

The finite element evaluations of crash problems can be extremely expensive (100+ CPU hours). Metamodels—approximations to the structural performance, built using FEA evaluations of a selected set of designs—are commonly used to reduce costs.

Accurate probabilistic results requires an accurate metamodel. A significant body of research exists on the application of metamodels to structural optimization and probabilistic evaluation. It has been shown that (i) the accuracy of a response surface is best improved by considering a smaller region of interest [16] and (ii) computing the variance (standard deviation) of the response is more demanding than the computation of the mean value [17].

Impact problems are distinguished from other structural problems by (i) noise in the results and (ii) very sparse sampling of the design space due to the high computational costs of, for example, a full vehicle crash simulation. The requirements dictate a metamodel that can separate the effects of the design parameters from the noise in the solution using a parsimonious number of structural evaluations. Using a small region of interest is a crucial first step because it reduces the curvature in the structural performance thereby allowing more structural evaluations to be used to estimate the noise. Quadratic response surfaces seem ideal, but in cases with extreme computational cost, linear response surfaces with an associated reduction in accuracy may be the only computationally feasible solution. Response surfaces built over a small region of interest, as used in this study, is the simplest and most robust metamodel formulation successfully meeting the requirements. The size of the region of interest must be selected carefully, because for very small regions of interest, it may be difficult to distinguish the effect of the design variables from the noise in the problem, while for large regions of interest it can be difficult to create an approximation to the potentially substantial amount of curvature in the structural performance. Should global approximations be essential and affordable, or should the response be very non-linear, then neural networks [18] and Kriging approximations [19] may be considered, provided that the robustness of the implementation with respect to noisy results is well understood.

3.1. Response surfaces as a projection on a predictable space

Consider a scalar response $\mathbf{y}$ dependent on the variable vector $\mathbf{x}$ through the relationship $y(\mathbf{x})$ including potential bifurcations, noise, and errors. We have

$$y = y(\mathbf{x})$$

where we want to approximate the relationship y using a response surface [20] $f(\mathbf{x})$ as

$$f(\mathbf{x}) = \sum_{i=0}^{p} a_i \varphi_i(\mathbf{x})$$

with a_i the coefficient for the basis function φ_i, and using p basis functions. The basis functions form a basis space for the changes in response that can be ascribed to the design variables, and are frequently chosen as the monomials $(1, x_1, \ldots, x_m, x_1^2, x_1x_2, \ldots, x_1x_m, \ldots, x_m^2)$ for the quadratic approximation using m variables. The coefficients $\mathbf{a}$ are computed as

$$\mathbf{a} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y}$$

which minimizes $\mathbf{r}^{\mathrm{T}}\mathbf{r}$, the square of the residuals. The basis function matrix is

$$X = [X_{i\alpha}] = [\varphi_\alpha(x_i)]$$

where the response is evaluated at $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ for a total of n experiments. The points in the design space where the response must be evaluated in order to create the response surface are selected using design of experiment techniques [20]. A number of experimental design methods are available [10]. For the problems considered in this study, techniques that can handle random variation are appropriate.

Variation in the responses, such as buckling, that cannot be described by this basis function space will be residuals, $r(\mathbf{x})$. We have therefore

$$y(\mathbf{x}) = f(\mathbf{x}) + r(\mathbf{x}) = P(y(\mathbf{x})) + R(y(\mathbf{x}))$$

with $\mathbf{f}^{\mathrm{T}}\mathbf{r} = 0$, $P(y(\mathbf{x}))$ the projection of the response onto the predictable space, and $R(y(\mathbf{x})) = R(P(y(\mathbf{x})))$ the projection onto the one-dimensional residual space.

The response variation and probabilities of events can be computed cheaply doing a Monte Carlo analysis considering the response surface $f(\mathbf{x})$, the variation of the variables, and the process variation as

$$y = f(\mathbf{x}) + \mathrm{N}(0, s^2)$$

where the process variation is approximated as the normal distribution N with a zero mean and variance s^2. Different methods of treating the process variation are considered in the next section.

3.2. Estimation of the process variation

The predicted values of the response are

$$\hat{\mathbf{y}} = \mathbf{X}(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y} = \mathbf{H}\mathbf{y}$$

from which the residuals can be computed, using the hat matrix $\mathbf{H}$, as

$$\mathbf{r} = \mathbf{y} - \hat{\mathbf{y}} = (\mathbf{I} - \mathbf{H})\mathbf{y}$$

The process variation is estimated from the sum of the residual mean squares as

$$s^2 = \frac{\sum_i^n r_i^2}{n - p}$$

with n the number of sampling points and p the number of basis functions. A normal distribution with zero mean and variance s^2 is usually assumed for the residuals. This assumption is not compulsory. For known behaviour of the structure, a different statistical distribution may be substituted; for example, a Bernoulli distribution describing a single bifurcation.

Assuming a distribution for the process variation is not essential. If it is believed that the residuals contain a representative population of all the possible values of the process variation, then the residuals can be bootstrapped [21] instead of using samples from a distribution describing the residuals. Bootstrapping the residuals is simply achieved by drawing a random sample from the residuals with replacements during the Monte Carlo evaluation. Bootstrapping the residuals is therefore desirable in cases where it is unclear which distribution to assume for the process variation, because bootstrapping can reproduce variation consisting of clusters of data as well as variation following a smooth statistical distribution.

Use of the PRESS residuals [20] is investigated for bootstrapping the process variation. A PRESS residual is computed using a response surface built using all the experimental points except the one used to compute the residual. The correlation between the response surface and the residuals is therefore reduced and outliers are easier to identify. The PRESS residuals can be computed using the diagonal terms of the hat matrix ($\mathbf{H}$) as

$$r_{(i)} = \frac{r_i}{1 - h_{ii}}$$

3.3. Error estimation

If the predictable space cannot completely describe the effect of the design variables then we have a bias error, $\delta(\mathbf{x}) = \eta(\mathbf{x}) - f(\mathbf{x})$, the difference between the chosen metamodel and the true effect of the design variables. The decomposition can be rewritten as

$$y = f(\mathbf{x}) + r(\mathbf{x}) = f(\mathbf{x}) + \delta(\mathbf{x}) + \varepsilon$$

The residuals therefore contain both the bias (fitting error) and the process variation ε containing any possible bifurcations; if the process variance is estimated using the residuals then the bias component is included in our estimate of the process variance. The bias error may be significant especially in the case of a large subregion in conjunction with high curvature of the response [22], in which case the estimate of $\sigma^2_{\text{process}}$ will usually be too large. For a model-independent estimate, replicate runs (multiple observations for the same design) are required, but is not always possible in practice, because a computer experiment, unlike most physical experiments, will always give exactly the same results if executed again using exactly the same setup. Given that the process variation should be overestimated, we estimate the upper bound on the process variation as the computed value and the lower bound as no process variation at all. Experience with a specific problem and the methodology is crucial for an accurate estimate of the process variation.

4. INSTABILITY VISUALIZATION

The proposed instability visualization methodology is simply the visualization of the process variation. The metamodels fit to the effects of the design variables but not to the instabilities. The nodal displacement residuals (process variation) are therefore composed of instability effects such as different buckling modes. The instabilities can therefore be viewed by viewing the process variation. In the following sections, we use the terms process variation, residuals, and outliers interchangeably, depending on context, because the process variation is estimated using the residuals as described earlier, while outliers are residuals indicative of bifurcations.

4.1. Visualization of unstable zones

A basic algorithm is

Step 1: For every node in the FE model

For every displacement component
- Fit a response surface to the nodal displacement of the node
- Collect the residuals of the response surface to the nodal displacement component
- Compute the standard deviation or range of the residuals (process variation)

Step 2: Save the residual standard deviation (process variation) or range from step 1 in a database suitable for visualization

Step 3: For all displacement components

Fringe plot the nodal results from step 2 on the model

Unstable areas in the structure will be clearly identifiable in step 3.

4.2. Visualization of instability cause–effect relationships

To see whether two events A and B are related, the covariance of the random variables Y_{A} and Y_{B} can be computed as

$$\mathrm{Cov}(Y_{\mathrm{A}}, Y_{\mathrm{B}}) = E[(Y_{\mathrm{A}} - \mu_{\mathrm{A}})(Y_{\mathrm{B}} - \mu_{\mathrm{B}})]$$

with $\mu_{\mathrm{A}} = E(Y_{\mathrm{A}})$, $\mu_{\mathrm{B}} = E(Y_{\mathrm{B}})$, and $E(Y) = 1/n \sum_{i=1}^{n} y_i$. The covariance can be difficult to assess because it is not scaled. The coefficient of correlation, which will vary between -1 and 1, can be used instead. The coefficient of correlation is

$$\rho = \frac{\mathrm{Cov}(Y_1, Y_2)}{\sigma_1 \sigma_2}$$

The coefficient of correlation of all the nodal displacement component residuals (process variation) with respect to the residuals (process variation) of a response can be plotted on the model using the algorithm given in the previous section. Evaluation of the coefficient of correlation with different displacement components may identify the instability mechanism; for example, an x-displacement coefficient of correlation of -1 would indicate that the area is always moving in the negative x-direction when the response value is larger than expected.

The number of experiments required for the correlation to be significant depends on the magnitude of the coefficient of correlation: a large coefficient of correlation requires only a small number of experiments, while, inversely, a small coefficient of correlation requires a large number of experiment points. If the coefficient of correlation is small, then it is cost-effective to consider it insignificant rather than to conduct additional experiments to investigate significance. Additionally, the industrial problems considered indicate that at least 30 numerical experiments should be planned for, because the use of fewer experiments may result in ambiguous plots for large, fairly noisy structural problems.

An alternative methodology is to use cluster analysis to trace a bifurcation back to its origin [5].

5. EXAMPLES

Three examples are used: the analytical problems clarify accuracy, the head impact model clarifies some structural mechanics details, and the occupant safety study shows applicability to large industrial models. The structures were evaluated using LS-DYNA [23] finite element analysis code and the LS-OPT design optimization code [24] was used to conduct the non-deterministic analysis.

5.1. Analytical examples

The analytical examples are used to investigate the accuracy of predicting the process variation and residuals for ten design variables and a varying number of experimental points. The process variation is estimated using the residuals; the difference between the residuals computed and the known process variation is therefore investigated. Also, a physical occurrence such as a bifurcation may manifest itself as an outlier amongst the residuals, and the residuals must therefore be computed as accurately as possible in order to investigate these physical occurrences.

The data considered have no deterministic component and consist only of process variation. The results are however computed as if there were a deterministic contribution, which allows the methodology to ascribe some of the process variation to deterministic variation, and accordingly allows us to evaluate how accurate the process variation can be estimated using the method. The accuracy of the method is established by monitoring the standard deviation of the process variation and the error of computing the individual residuals. The error of computing the residuals is quantified using the mean and standard deviation of the error of estimating each residual as $\text{Error}(r_i) = |\varepsilon - r_i|$ with ε the correct value of the residual (the process variation) and r_i the value of the computed residual. The correct values of the process variation ε and accordingly the correct value of each individual residual are simply that of the response values entering the computations. Note that the computation considers the computation of the residuals of the residuals (higher order residuals)—the correct values of the residuals (process variation) are known, and by comparing these with the computed residuals, one obtains the residuals of computing the residuals, or rather the error of computing the residuals (process variation).

The experimental designs used in the following two experiments are D-optimal [20] for linear approximations with the larger experimental designs obtained by augmenting the smaller experimental designs. The design variables have lower bounds of -1 and upper bounds of 1. A linear response surface is fitted to the results to simulate the effect of incorrectly assigning process variation to the design variables.

In the first test a single outlier of value 1 is considered, with all other points having no process variation (of value 0). The results are as given in Table I. Of particular interest for this study is the error of estimating the residuals. The table shows that for a very small number of experiments, a single outlier may cause significant errors in the residuals. It can be seen that the PRESS residuals overestimate the residuals for a small number of experiments signifying that the detrimental effect of using a smaller number of experimental points to build the response surface is larger than the benefit gained by using the PRESS residuals.

In the second test the results are a Bernoulli distribution with a 50% chance of having a value of 0 or 1. The results are as given in Table II. Both the error of estimating the process variation and the error of estimating the residuals are of interest for this study. The table indicates that the residuals can be a good predictor of the process variation. But, for a small number of experiments, both the estimation of the process variance and the estimation of outlier distances are problematic.

Table I. Process variation of single outlier.

	σ_{process}		Residual estimation error	
Experiments	Actual	Estimated*	Mean*	σ*
15	0.26	0.26 (0.46)	0.13 (0.28)	0.21 (0.33)
20	0.22	0.29 (0.32)	0.11 (0.17)	0.14 (0.16)
30	0.18	0.22 (0.24)	0.08 (0.10)	0.08 (0.10)
50	0.14	0.16 (0.16)	0.05 (0.06)	0.04 (0.05)
100	0.10	0.11 (0.11)	0.03 (0.03)	0.02 (0.02)
500	0.04	0.05 (0.05)	0.01 (0.01)	0.00 (0.00)

*The standard method of computing the residuals were used to compute the first value, while the PRESS residuals were used to produce the values in brackets.

Table II. Process variation of Bernoulli distribution.

	$\sigma_{process}$		Residual estimation error	
Experiments	Actual	Estimated*	Mean*	σ*
15	0.47	1.11 (1.37)	0.33 (0.86)	0.23 (0.67)
20	0.50	0.68 (0.76)	0.30 (0.44)	0.24 (0.37)
30	0.50	0.54 (0.56)	0.29 (0.34)	0.21 (0.24)
50	0.49	0.52 (0.53)	0.22 (0.23)	0.14 (0.17)
100	0.50	0.50 (0.50)	0.17 (0.18)	0.13 (0.14)
500	0.50	0.51 (0.51)	0.04 (0.04)	0.03 (0.03)

*The standard method of computing the residuals were used to compute the first value, while the PRESS residuals were used to produce the values in brackets.

Both tests show that if it is not feasible to assume a statistical distribution for the process variation then the PRESS residuals can be bootstrapped instead. The PRESS residuals tend to overestimate each individual residual and accordingly the process variation, especially for a small number of experiments.

5.2. Head impact problem

We consider the problem of a Free Motion Headform impacting an A-pillar as shown in Figure 2. The mesh is parameterized using the TrueGrid preprocessor [25]. We consider two variables: the angle of the impact and the rib stiffener height of the pillar padding. We investigate the variance on the Head Injury Criterion, HIC-d $= 166.4 + 0.75466 * \text{HIC15}$. The HIC15 Head Injury Criterion is computed by finding the 15 ms interval with the maximum average head acceleration and averaging the acceleration over this interval.

Firstly we investigate the problem using a parametric study in which we vary only one of the variables at a time. The results for varying the angle of impact are as shown in Figure 3. From the figure it is clear that the problem has some noise and it suggests that the variables have a mostly linear effect on the HIC response. The results for varying only the rib height variable are similar. If clear clustering can be distinguished, then state-selecting metamodels [26] or a Bernoulli distribution describing the process variation can be considered. But this is not the case here and a normal distribution is assumed for the process variation.

5.2.1. Probabilistic investigation. The statistics of the HIC-d response was investigated using (i) a Monte Carlo analysis and (ii) a quadratic response surface. The angle of impact is taken to be 15° with a 2.5 percent standard deviation, normally distributed, while the rib height is 12.5 mm with a 5 percent standard deviation, also normally distributed. These values allow the angle of impact variation, rib height variation, and process variation to have a roughly equal influence on the HIC-d variation.

The Monte Carlo analysis was done using 500 FEA evaluations and a Latin Hypercube experimental design. The results from this Monte Carlo analysis along with their confidence bounds are given in Table III.

A quadratic response surface was computed using the FEA results from the 500 points Monte Carlo analysis. The adjusted R^2 indicator of quality of fit was 0.817 and the process

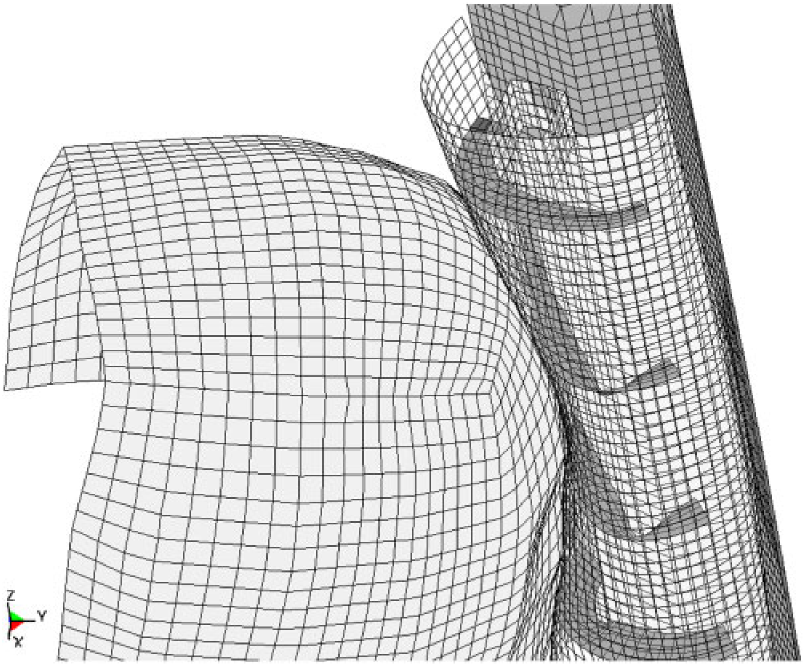

Figure 2. Head impact FE model. A Free Motion Headform impacts an A-pillar. The pillar padding is internally stiffened with ribs. Both the angle of impact and the rib stiffener height are varied during the analysis.

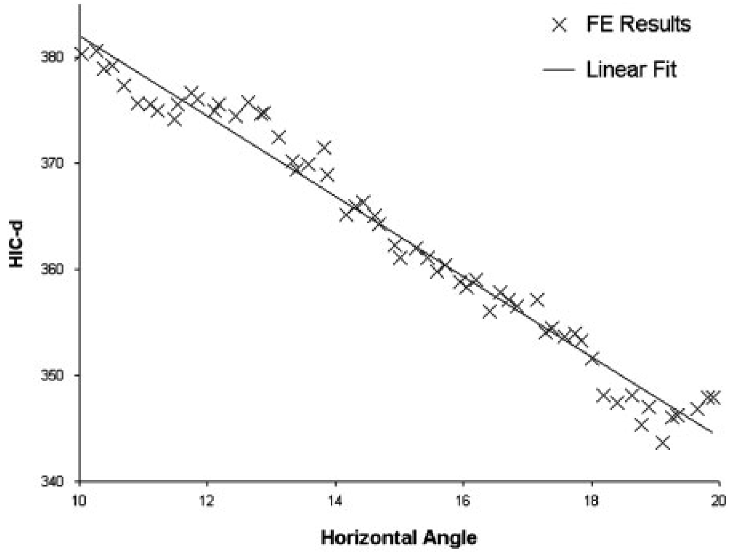

Figure 3. HIC-d—horizontal angle variation. The horizontal angle variation has a linear effect of the HIC-d response, and some noise in the response can be observed. The noise does not exhibit a clear clustering behaviour.

Table III. Comparison of Monte Carlo and response surface results.

	Monte Carlo			Quadratic response surface	
	Lower bound*	Value	Upper bound*	$\sigma^2_{total} = \sigma^2_{deterministic} + \sigma^2_{process}$ †	$\sigma^2_{total} = \sigma^2_{deterministic}$
HIC-d Mean		363		363 (363)	363
HIC-d Std dev	1.89	2.14	2.43	2.06 (2.06)	1.85
P[HIC-d>357.5]	1.000	1.000	1.000	0.999 (0.998)	1.000
P[HIC-d>360.0]	0.931	0.95	0.969	0.946 (0.947)	0.967
P[HIC-d>362.5]	0.556	0.600	0.644	0.622 (0.622)	0.633
P[HIC-d>365.0]	0.164	0.200	0.236	0.187 (0.186)	0.161
P[HIC-d>367.5]	0.017	0.034	0.050	0.024 (0.024)	0.159
P[HIC-d>370.0]	0.000	0.000	0.000	0.001 (0.001)	0.001

*95% confidence interval.
†A normal distribution was assumed for the process variation to compute the first value, while the PRESS residuals were bootstrapped to produce the values in brackets.

variation is estimated to have a standard deviation of 0.918 (0.2% of the mean value). Results are computed from the metamodel using a Monte Carlo analysis of 10^6 points. The results from this metamodel-based Monte Carlo analysis are given in Table III along with the values from the pure Monte Carlo analysis.

The results for the probabilistic evaluation are also given in Table III. The table shows that if the variance of the response is estimated using only the deterministic variation as computed using the quadratic response surface, then the resulting value underestimate the Monte Carlo (benchmark) value. The process variation (estimated using the residuals from the response surface fit) must be included into the computations in order to correlate with the Monte Carlo results.

Table III also shows that the two different methods of estimating the process variation, assuming a normal distribution and bootstrapping the PRESS residuals, gave practically identical results.

5.2.2. Bifurcation investigation. Significant residuals were obtained from the response surface fit. This lack of fit is judged not to be due to the choice of metamodel (bias error)—the preliminary study considering only a single variable showed that a quadratic response surface should fit well over the region of interest. Bifurcations (buckling) or poor modelling may be responsible for these residuals; in this section we show how the process variation can be investigated to identify bifurcations.

The proposed instability visualization method is used. We approximate the displacement of every node in the model using quadratic responses surfaces, from which the process variations of the displacements are computed. Problem areas in the structure are visible when the process variation is plotted on the model as demonstrated in Figure 4, which shows that the displacements of one rib edge are not explained by the variation of the design variables.

Examining the identified region for the FEA experiments associated with the maximum and minimum outlier highlights the different buckling modes are as shown in Figure 5.

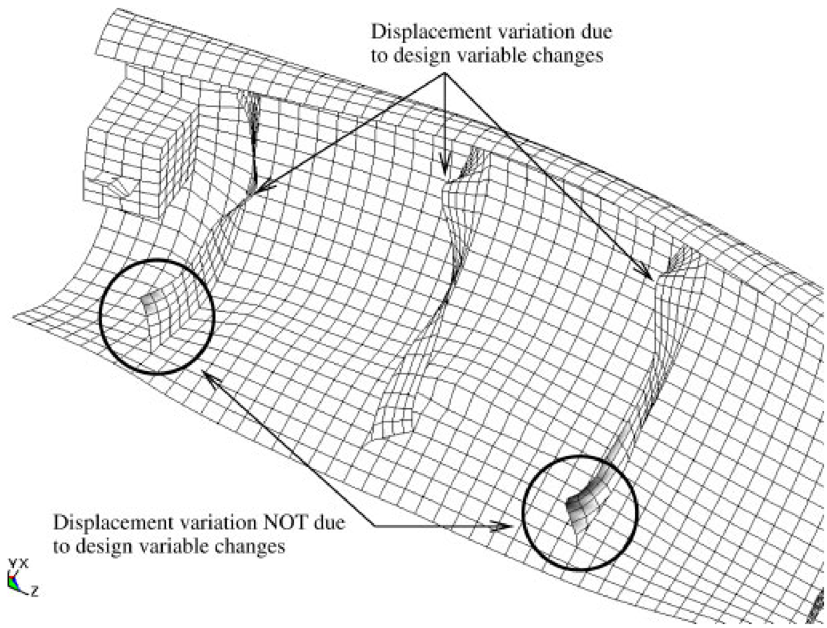

Figure 4. Fringe plot of the z-displacement process variation. The dark areas, having high values of the process variation, indicate that the changes in the z-displacements values in these areas cannot be associated with changes in the design variable values. Stability problems are therefore suspected in these dark areas.

5.3. Occupant simulation model

A sled model test for the validation of occupant simulation is used to evaluate the technology for large finite element models and a large number of variables. The model has 123 000 elements and a simulation takes 9.5 h on 16 processors. The overlay plot of the model in Figure 6 shows the bifurcation of the right-hand movement.

Ten variables were considered using the following four setups:

1. *Convergence of the number of points*: The full set of variables is used and the number of experimental points is varied over a number of studies using this setup.
2. *Invariance with respect to the region of interest*: The range of the design variables is increased and the number of design variables is decreased.
3. *Control setup for the computation of the derivatives*: Only one parameter is varied with the number of points selected in order to compare the process variation with setup four.
4. *Control setup for the process variation computation*: The structural parameters are not varied; instead process parameters such as the number of CPUs and contact parameters are varied to estimate the process variation.

Scaled values of the design variables for the four setups are given in Table IV. For every setup a number of studies are considered with the resulting eleven studies as given in Table V.

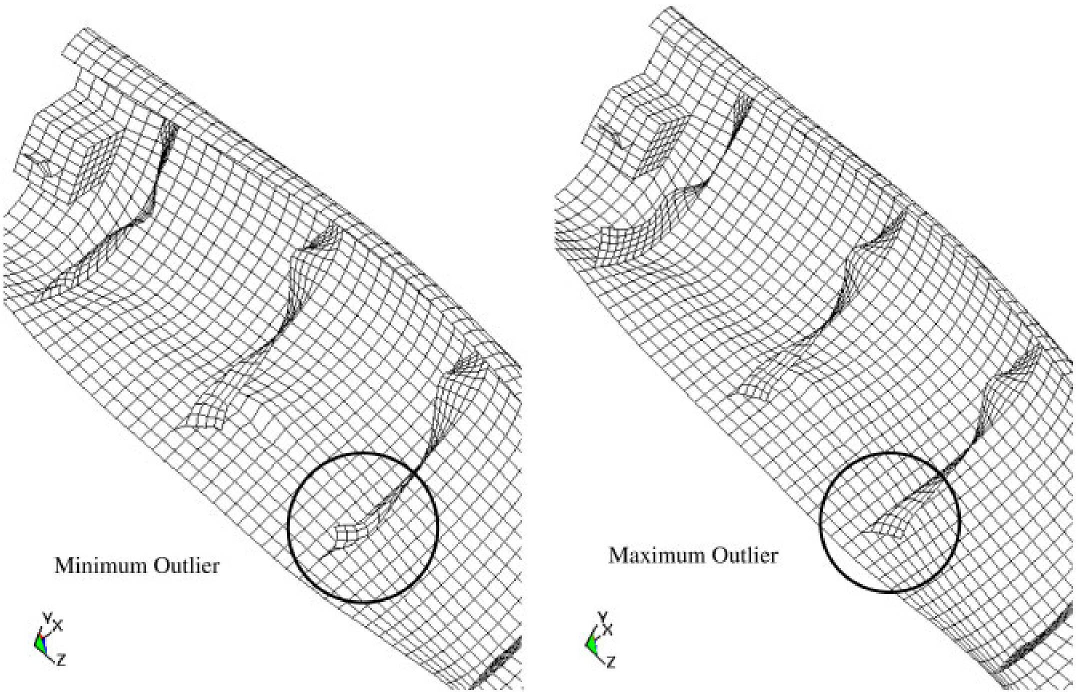

Figure 5. Buckling modes of the maximum and minimum outlier. The different buckling modes were found by identifying an area having stability problems using Figure 4 and subsequently examining the FE jobs producing the minimum and maximum value of the z-displacement residuals in this area.

The responses monitored are the head impact criterion (HIC36), the chest intrusion, the shoulder belt force, the pelvis belt force, the chest acceleration, and the pelvis acceleration. The HIC36 Head Injury Criterion is computed by finding the 36 ms interval with the maximum average head acceleration and averaging the acceleration over this interval.

The deterministic and the process x-displacement variation of the FE model are shown in Figure 7. The head displacement is quite predictable, because it has a high value of the deterministic variation and a low process variation value. The difficulty of predicting the movement of the extremities of the occupant model and the bifurcation of the hand movement is clearly visible from the process variation plot.

Figure 8 shows the coefficient of correlation of the pelvis belt force residuals with the nodal z-displacement residuals. The coefficient of correlation is high for the seatbelt; the plot therefore alleges that slip-stick conditions of the seatbelt are causing the process variation of the belt pelvis force.

The results from the studies are summarized in the graphs in Figures 9–11. The mean response computed from the response surfaces are compared with the results at the nominal design in Figure 9. In Figure 10 we show the derivatives of the responses with respect to the sled acceleration. The 'correct' value of the derivative given is the average from studies L3-33 and Q3-33, which were in very close agreement. The accuracies of the derivatives

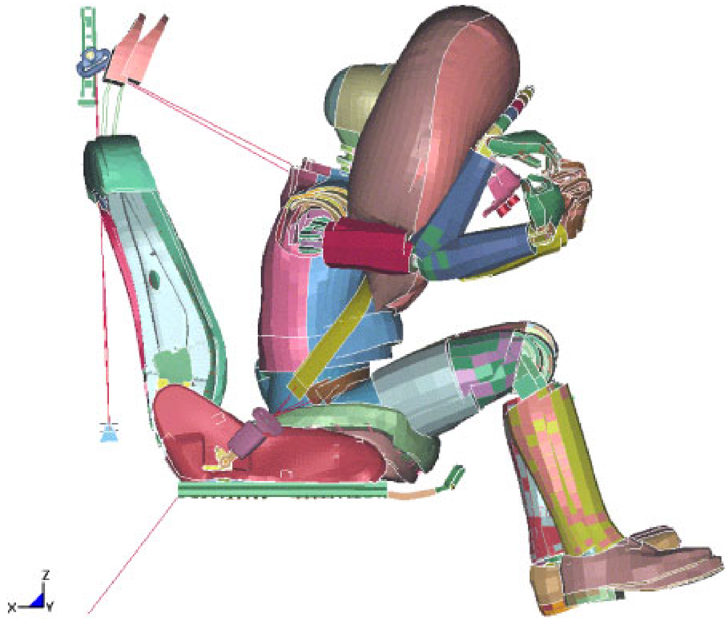

Figure 6. Overlay plot of two occupant simulation runs. Note the bifurcation of the right-hand movement.

Table IV. Occupant simulation variable setups.

Variable	Nominal value	Setup 1 σ	Setup 1 Range*	Setup 2 σ	Setup 2 Range*	Setup 3 σ	Setup 3 Range*
Sled acceleration	1.00	0.03	0.95–1.05	0.05	0.90–1.10	0.05	0.90–1.10
Slip ring 1 friction	1.00	0.25	0.50–1.50	0.50	0.00–2.00		
Slip ring 2 friction	1.00	0.25	0.50–1.50	0.50	0.00–2.00		
Seat belt pre-tension	1.00	0.04	0.91–1.09	0.09	0.82–1.18		
Force limit retractor	1.00	0.06	0.89–1.11	0.11	0.78–1.22		
Steering wheel rotation	−1.00	0.05	−1.10 – −0.9	0.10	1.19 – −0.81		
Dashboard x-translation	0.00	0.50	−1.00–1.00	1.00	−2.00–2.00		
Dashboard z-translation	0.00	0.50	−1.00–1.00				
Airbag mass flow	1.00	0.05	0.90–1.10				
Young's modulus dashboard Aluminum	1.00	0.05	0.90–1.10				

Note: Setup 4 contained computational parameters only: the number of CPUs, the computer manufacturer, a soft constraint formulation, a non-structural beam thickness, a contact penalty, the contact bucket sort frequency, and the offset of some non-structural nodes.

*For the construction of the response surface, the variables are allowed to vary over a given range. This range affects only the construction of the response surfaces and not the probabilistic computations.

Table V. Occupant simulation studies.

Study	Setup	Number of experiments	Experimental selection	Design variables and ranges	Response surface order
L1-30	1	30	D-optimal for linear response surface [19]	Setup 1	Linear
L1-60	1	60		Setup 1	Linear
L1-90	1	90		Setup 1	Linear
L1-120	1	120		Setup 1	Linear
L2-30	2	30	D-optimal for linear response surface [19]	Setup 2	Linear
L2-60	2	60		Setup 2	Linear
Q1-90	1	90	D-optimal for linear response surface [19]	Setup 1	Quadratic
Q1-120	1	120		Setup 1	Quadratic
L3-33	3	33	Space filling [27]	Setup 3	Linear
Q3-33	3	33		Setup 3	Quadratic
Pure noise	4	30	Random	Setup 4	None

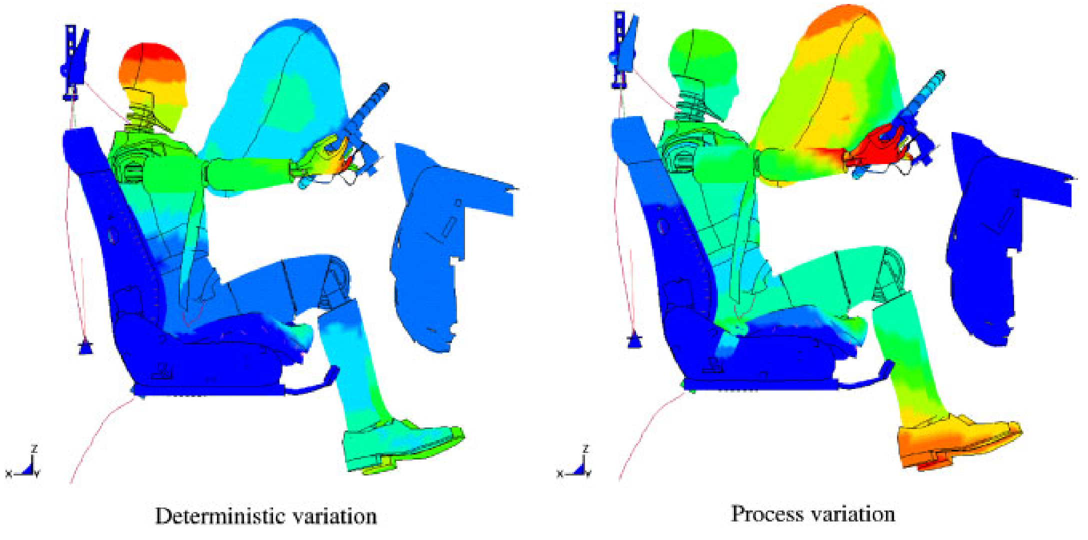

Figure 7. Occupant x-displacement variations. Red areas have the highest x-displacement variation; blue areas, the least. The deterministic and process variations are not to the same scale. Note the predictability of the head displacement compared to the difficulty of predicting the right-hand movement.

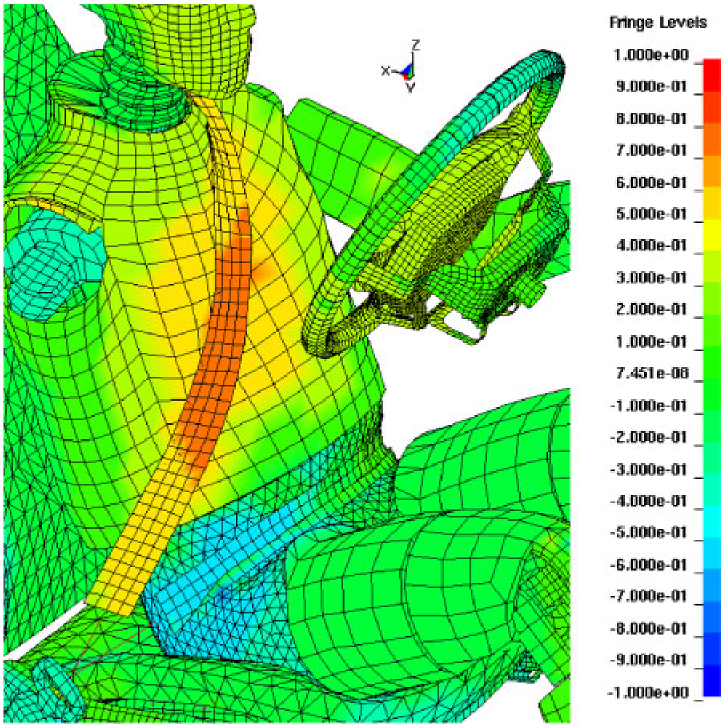

Figure 8. Correlation of pelvis belt force and nodal z-displacement outliers. The figure shows that the seatbelt z-displacement outliers are correlated with the pelvis belt force outliers—stick-slip conditions of the seatbelt may therefore be causing the pelvis belt force variation.

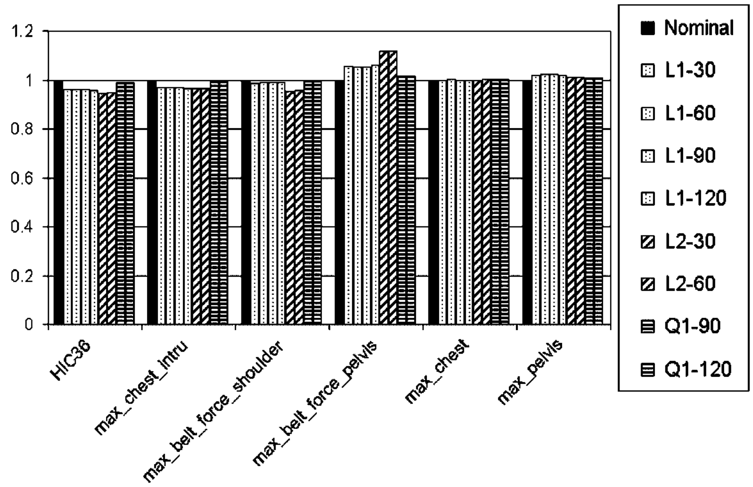

Figure 9. Response mean scaled with nominal response. The mean differs slightly between experimental setups and response surface order.

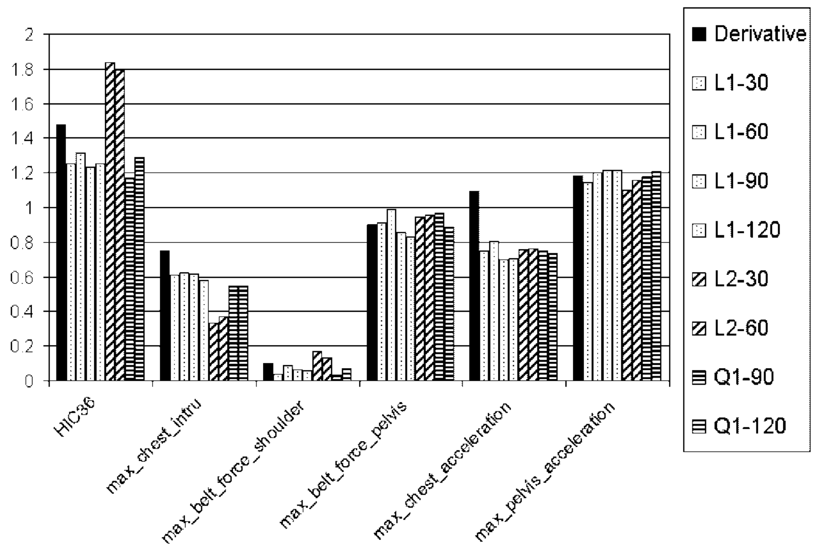

Figure 10. Response derivatives with respect to sled acceleration. The derivatives are scaled with the mean response and their accuracies are indicative of the accuracy of the deterministic variation. Differences of up to 25% with respect to the 'correct' derivative can be noted, presumably due to noise and curvature of the responses.

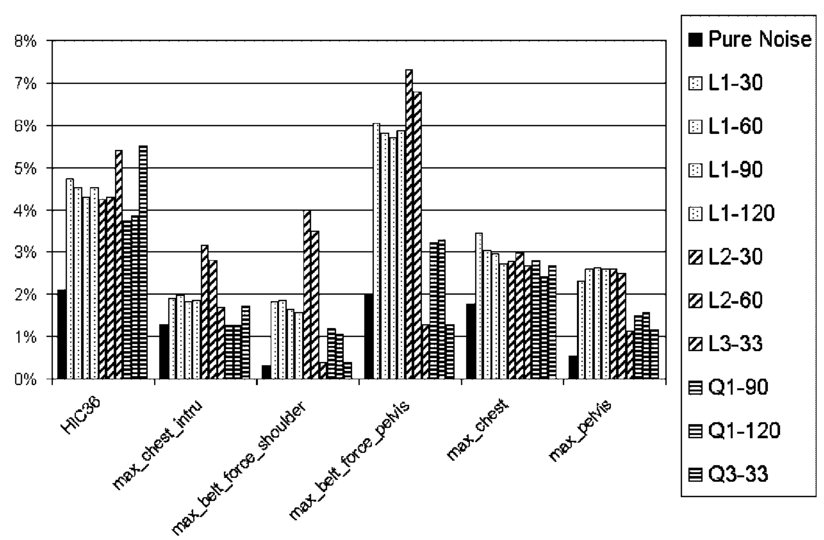

Figure 11. Standard deviation of the estimated process variation scaled with the nominal value of the response. The results are the same order of magnitude. The process variation is overestimated by the response surfaces, presumably due to bias error.

are indicators of the accuracy of the deterministic variation—for linear response surfaces, the deterministic variation is obtained by summing the squares of the products of the derivatives with the standard deviation of the variables. The estimates to the process variance are shown in Figure 11. The results from the eleven studies indicate good agreement of the mean, derivatives, and process variation estimates for the different response surfaces. The process variation and the response surface estimate have similar orders of magnitude, but the response surfaces consistently overestimate the process variation, presumably due to the bias error (additional investigation of this problem [28] has shown that feed forward neural networks [18] provides an estimate of the process variation in close agreement to the pure noise estimate). The quadratic response surfaces perform slightly better than the linear response surfaces, but the corresponding cost increase (additional FEA runs) is difficult to justify.

The HIC36, the pelvis belt force, and the chest acceleration have the highest process variances. Assigning a specific cause to a process variance is not easy: the potential sources—the hand movement bifurcation shown, stick-slip conditions of the seat belt, and effects from the airbag—are acting simultaneously. In particular, the variation of the HIC36 criterion will require careful study to be understood—the HIC36 criterion is a complex computation dependent on the head acceleration, influenced by both the airbag deployment process variation and stick-slip of the seat belt; but the noise in the HIC36 postprocessing computations may also be responsible. Stick-slip conditions of the seat belt are a likely cause of variance of the pelvis belt force.

Linear response surfaces computed from 30 FE runs seem suitable for this structural problem. More complex metamodels, associated with a larger number of experimental points, will be required only for regions of the design space known to have a larger curvature in the response.

6. CONCLUSIONS

The process variation—variation not associated with a change in the variables, though intrinsic to the structural mechanics of the event—contributes to the variation of the responses. This uncertainty can be used in conjunction with the metamodels when computing probabilities of events. But if the process variation is not associated with the physics of the problem, then it should not be incorporated into the reliability computations and a metamodel filtering out the process variation should be used.

Using metamodels to compute the response variation is attractive for reasons of cost and distinguishing between bifurcation driven changes in the results and changes in the results due to changes in the design variables.

The residuals (outliers) obtained from the metamodel fitting procedure are not always due to an inadequate metamodel and may indicate the presence of different physical solutions.

Designing for uncertainty requires a focus on the eigenvalues (stability) of the structure in addition to considering the design variable values.

The proposed stability visualization methodology, possibly the principal contribution of this study, was shown to be effective for industrial problems. Computing the process variation accurately in industrial problems, on the other hand, is challenging and requires care in both the modelling and the analysis of the problem.

The process variation can be reduced by redesigning the structure if the process variation is physical, and by careful modelling if non-physical.

To reduce the bias component of the residuals, more advanced meta-models may be considered. Response surfaces are used here for efficiency and clarity.

ACKNOWLEDGEMENTS

DaimlerChrysler Commercial Vehicles sponsored the evaluation of the methodology on industrial models. We are indebted to Paul du Bois for a discussion regarding the mechanics of these problems. Suri Balasubramanyam and Mike Burger prepared the head impact model.

REFERENCES

1. Robinson DG. A survey of probabilistic methods used in reliability, risk and uncertainty analysis: analytical techniques I. *SAND98-1189*. Sandia National Laboratories: Albuquerque, NM, 1998.
2. Wojtkiewicz SF, Eldred MS, Field RV, Urbina A, Red-Horse JR. Uncertainty quantification in large computational engineering models. *AIAA-2001-1455*, *Proceedings of the 42nd AIAA/ASME/ASCE/AHS/ASC Structures*, *Structural Dynamics*, *and Materials Conference*, Seattle, WA, 16–19 April 2001.
3. Kale A, Acar E, Haftka RT, Stroud WJ. Why are airplanes so safe structurally? Effects of various safety measures on structural safety of aircraft. *AIAA-2004-1629*, *Proceedings of the 45th AIAA/ASME/ASCE/AHS/ASC Structures*, *Structural Dynamics*, *and Materials Conference*, Palm Springs, CA, 19–22 April 2004.
4. Blumhardt R. FEM—crash simulation and optimization. *International Journal of Vehicle Design* 2001; **26**:331–347.
5. Thole A-A, Liguan M. Reasons for scatter in crash simulation results. *Proceedings of the 4th European LS-DYNA Conference*, Ulm, Germany, 2003.
6. Belytschko T, Mish K. Computability in non-linear solid mechanics. *International Journal for Numerical Methods in Engineering* 2001; **52**:3–21.
7. Missoum S, Chaabane SH, Sudret B. Handling bifurcations in the optimal design of transient dynamic problems. *Proceedings of the 45th AIAA/ASME/ASCE/AHS/ASC Structures*, *Structural Dynamics*, *and Materials Conference*, Palm Springs, CA, 19–22 April 2004.
8. Crisfield MA, Peng X. Instabilities induced by coarse meshes for a nonlinear shell problem. *Engineering Computations* 1996; **13**:110–114.
9. Goel T, Mack Y, Haftka RT, Shyy W, Queipo NV. Interaction between grid and design space refinement for bluff body-facilitated mixing. *Proceedings of the 43rd AIAA Aerospace Sciences Meeting and Exhibit*, Reno, NV, 8–13 January 2005.
10. Giunta AA, Wojtkiewicz SF, Eldred MS. Overview of modern design of experiments methods for computational simulations. *AIAA-2003-0649*, *Proceedings of the 41st AIAA Aerospace Sciences Meeting and Exhibit*, Reno, NV, 6–9 January 2003.
11. Schoofs AJG. Experimental design and structural optimization. *Ph.D. Thesis*, Technische Universiteit Eindhoven, 1987.
12. Schenk CA, Schuëller GI. Buckling analysis of cylindrical shells with random geometric imperfections. *International Journal of Non-Linear Mechanics* 2003; **38**:1119–1132.
13. Riha DS, Thacker BH, Hall DA, Auel TR, Pritchard SD. Capabilities and applications of probabilistic methods in finite element analysis. *Proceedings of the Fifth ISSAT International Conference on Reliability and Quality in Design*, Las Vegas, NV, 1999.
14. Tu J, Choi KK. Design potential concept for reliability-based design optimization. *Technical Report R99-07*, Center for Computer Aided Design and Department of Mechanical Engineering, College of Engineering, University of Iowa, 1999.
15. Thacker BH, Riha DS, Millwater HR, Enright MP. Errors and uncertainties in probabilistic engineering analysis. *AIAA 2001-1239*, *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 42nd Structures*, *Structural Dynamics*, *and Materials* (*SDM*) *Conference*, Seattle, WA, 16–19 April 2001.
16. Roux WJ, Stander N, Haftka RT. Response surface approximations for structural optimization. *International Journal for Numerical Methods in Engineering* 1998; **42**:517–543.
17. Jin R, Du X, Chen W. The use of metamodeling techniques for optimization under uncertainty. *Structural and Multidisciplinary Optimization* 2003; **25**:99–116.

18. Bishop CM. *Neural Networks for Pattern Recognition*. Oxford University Press: Oxford, 1995.
19. Simpson TW. A concept exploration method for product family design. *Ph.D. Thesis*, Georgia Institute of Technology, 1998.
20. Myers RH, Montgomery DC. *Response Surface Methodology*. Wiley: New York, 1995.
21. Mooney CZ, Duval RD. *Bootstrapping. A Nonparametric Approach to Statistical Inference*. Sage Publications: Newbury Park, CA, 1993.
22. Draper NR, Smith H. *Applied Regression Analysis* (2nd edn). Wiley: New York, 1981.
23. Hallquist JO. *LS-DYNA Theoretical Manual*. Livermore Software Technology Corporation: Livermore, CA, 1998.
24. Stander N, Roux WJ, Eggleston T, Craig KJ. *LS-OPT User's Manual Version 2.2*. Livermore Software Technology Corporation: Livermore, CA, 2004.
25. *TrueGrid Manual*. XYZ Scientific Applications, Inc.: Livermore, CA, 2000.
26. Meckesheimer M, Barton RR, Simpson TW, Limayem F, Yannou B. Metamodeling of combined discrete/continuous responses. *AIAA Journal* 2001; **39**:1955–1959.
27. Johnson M, Moore L, Ylvisaker D. Minimax and maximin distance designs. *Journal of Statistical Planning and Inference* 1990; **26**:131–148.
28. Günther F, Müllerschön H, Roux WJ. Robust design for crash at DaimlerChrysler commercial vehicles CAE. *Proceedings of 3rd German LS-DYNA Conference*, Bamberg, Germany, 2004.

